

Leonard triples of q -Racah type

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Abstract

Let \mathbb{F} denote a field, and let V denote a vector space over \mathbb{F} with finite positive dimension. Pick a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$, and let A, B, C denote a Leonard triple on V that has q -Racah type. We show that there exist invertible W, W', W'' in $\text{End}(V)$ such that (i) A commutes with W and $W^{-1}BW - C$; (ii) B commutes with W' and $(W')^{-1}CW' - A$; (iii) C commutes with W'' and $(W'')^{-1}AW'' - B$. Moreover each of W, W', W'' is unique up to multiplication by a nonzero scalar in \mathbb{F} . We show that the three elements $W'W, W''W', WW''$ mutually commute, and their product is a scalar multiple of the identity. A number of related results are obtained.

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1 Introduction

This paper is about a linear algebraic object called a Leonard triple [8]. Before describing this object, we first describe a more basic object called a Leonard pair [37]. Let \mathbb{F} denote a field, and let V denote a vector space over \mathbb{F} with finite positive dimension. Let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of the \mathbb{F} -linear maps from V to V . According to [37, Definition 1.1], a Leonard pair on V is an ordered pair of maps in $\text{End}(V)$ such that for each map, there exists a basis of V with respect to which the matrix representing that map is diagonal and the matrix representing the other map is irreducible tridiagonal. As explained in [37, Appendix A], the Leonard pairs provide a linear algebra interpretation of a theorem of Doug Leonard [31], [3, p. 260] concerning the q -Racah polynomials and their relatives in the Askey scheme. The Leonard pairs are classified up to isomorphism [37, Theorem 1.9] and described further in [33, 34, 38–41, 43]. For a survey see [42]. In [8] Brian Curtin introduced the concept of a Leonard triple as a natural generalization of a Leonard pair. According to [8, Definition 1.2], a Leonard triple on V is a 3-tuple of maps in $\text{End}(V)$ such that for each map, there exists a basis of V with respect to which the matrix representing that map is diagonal and the matrices representing the other two maps are irreducible tridiagonal.

We give some background on Leonard triples. Their study began with Curtin's comprehensive treatment of a special case, said to be modular. By [8, Definition 1.4] a Leonard triple on V is modular whenever for each element of the triple there exists an antiautomorphism of $\text{End}(V)$ that fixes that element and swaps the other two elements of the triple. In [8, Section 1] the modular Leonard triples are classified up to isomorphism. The paper [9] gives a

natural correspondance between the modular Leonard triples, and a family of Leonard pairs said to have spin [9, Definition 1.2]. The spin Leonard pairs describe the irreducible modules for the subconstituent algebra of a distance-regular graph whose Bose-Mesner algebra contains a spin model [6, 10]. We will say more about modular Leonard triples shortly. The general Leonard triples have recently been classified up to isomorphism, via the following approach. Using the eigenvalues one breaks down the analysis into four special cases, called q -Racah, Racah, Krawtchouk, and Bannai/Ito [7, 20]. The Leonard triples are classified up to isomorphism in [26] (for q -Racah type); [12] (for Racah type); [29] (for Krawtchouk type); [24] (for Bannai/Ito type and even diameter); [21] (for Bannai/Ito type and odd diameter). Additional results on Leonard triples can be found in [19, 27, 30, 33, 44] (for q -Racah type); [1, 32, 36] (for Racah type); [2, 35] (for Krawtchouk type); [5, 11, 14, 15, 23, 25, 45] (for Bannai/Ito type). We mention two attractive families of Leonard triples, said to be totally bipartite or totally almost-bipartite (abipartite). The totally bipartite Leonard triples of q -Racah type are described in [22]; these are closely related to the finite-dimensional irreducible modules for the algebra $U_q(\mathfrak{so}_3)$ [16–18]. The totally bipartite/abipartite Leonard triples of Bannai/Ito type are described in [5].

Turning to the present paper, we consider a Leonard triple A, B, C on V that has q -Racah type. In order to motivate our results, assume for the moment that A, B, C is modular. By [9, Corollary 2.6] there exist invertible W, W', W'' in $\text{End}(V)$ such that (i) $AW = WA$ and $BW = WC$; (ii) $BW' = W'B$ and $CW' = W'A$; (iii) $CW'' = W''C$ and $AW'' = W''B$. Moreover by [9, Lemma 3.6] each of W, W', W'' is unique up to multiplication by a nonzero scalar in \mathbb{F} . By [8, Lemma 9.3] there exists an invertible $P \in \text{End}(V)$ such that each of $W'W, W''W', WW''$ is a scalar multiple of P . Moreover

$$P^{-1}AP = B, \quad P^{-1}BP = C, \quad P^{-1}CP = A$$

and P^3 is a scalar multiple of the identity. We now summarize our results. Dropping the modular assumption, we show that there exist invertible W, W', W'' in $\text{End}(V)$ such that (i) A commutes with W and $W^{-1}BW - C$; (ii) B commutes with W' and $(W')^{-1}CW' - A$; (iii) C commutes with W'' and $(W'')^{-1}AW'' - B$. Moreover each of W, W', W'' is unique up to multiplication by a nonzero scalar in \mathbb{F} . We show that the three elements $W'W, W''W', WW''$ mutually commute, and their product is a scalar multiple of the identity. Define $\overline{A} = W^{-1}BW - C$ and similarly define $\overline{B}, \overline{C}$. By construction \overline{A} commutes with A ; in fact \overline{A} is a polynomial in A and we describe this polynomial in several ways. We describe what happens if one of A, B, C is conjugated by one of $W^{\pm 1}, (W')^{\pm 1}, (W'')^{\pm 1}$; the result is a polynomial of degree at most 2 in $A, B, C, \overline{A}, \overline{B}, \overline{C}$. We also describe what happens if one of A, B, C is conjugated by one of $W^{\pm 2}, (W')^{\pm 2}, (W'')^{\pm 2}$; the result is a polynomial of degree at most 3 in A, B, C . We indicate how conjugation by $W^{\pm 2}, (W')^{\pm 2}, (W'')^{\pm 2}$ is related to the Lusztig automorphisms recently discovered by Baseilhac and Kolb [4]. Using some basic hypergeometric series identities, we express each of $W^{\pm 1}, W^{\pm 2}$ as a polynomial in A . We mentioned above that $W'W, W''W', WW''$ mutually commute. To obtain this result we show that (i) each of $W'W, W''W', WW''$ commutes with $A + B + C$; (ii) the subalgebra of $\text{End}(V)$ generated by $A + B + C$ contains every element of $\text{End}(V)$ that commutes with $A + B + C$. Near the end of the paper we show that $\overline{A} = \overline{B} = \overline{C} = 0$ if and only if A, B, C is modular, and in this case we recover the results of Curtin mentioned above.

This paper is organized as follows. Sections 2, 3 contain preliminaries. In Section 4 we review some basic facts about Leonard triples, and in Section 5 we consider the Leonard triples of q -Racah type. Section 6 contains some trace formulae that will be used later in the paper. In Sections 7, 8 we introduce the elements W , W' , W'' and \overline{A} , \overline{B} , \overline{C} . Sections 9, 10 are about conjugation. In Section 11 we express the elements $W^{\pm 1}$, $W^{\pm 2}$ as polynomials in A . In Sections 12, 13 we show that $W'W$, $W''W'$, WW'' mutually commute, and their product is a scalar multiple of the identity. Section 14 is about the case $\overline{A} = \overline{B} = \overline{C} = 0$.

2 Preliminaries about the eigenvalues

We now begin our formal argument. Throughout this section the following notation and assumptions are in effect. Fix nonzero $a, q \in \mathbb{F}$ such that $q^4 \neq 1$. Fix an integer $d \geq 0$, and define

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i} \quad (0 \leq i \leq d). \quad (1)$$

Lemma 2.1. *For $0 \leq i, j \leq d$,*

$$\theta_i - \theta_j = (1 - q^{2j-2i})(aq^{2i-d} - a^{-1}q^{d-2j}).$$

Proof. Use (1). □

Lemma 2.2. *The scalars $\{\theta_i\}_{i=0}^d$ are mutually distinct, if and only if the following hold:*

- (i) $q^{2i} \neq 1$ for $1 \leq i \leq d$;
- (ii) a^2 is not among $q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}$.

Proof. Use Lemma 2.1. □

Lemma 2.3. *For $1 \leq i \leq d$,*

$$\frac{q\theta_i - q^{-1}\theta_{i-1}}{q^2 - q^{-2}} = aq^{2i-d-1}, \quad \frac{q\theta_{i-1} - q^{-1}\theta_i}{q^2 - q^{-2}} = a^{-1}q^{d-2i+1}.$$

Proof. Use (1). □

Corollary 2.4. *For $0 \leq i, j \leq d$ such that $|i - j| = 1$,*

$$\frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}} \frac{q\theta_j - q^{-1}\theta_i}{q^2 - q^{-2}} = 1.$$

Proof. Use Lemma 2.3. □

Lemma 2.5. *For $0 \leq i \leq d$ pick $0 \neq t_i \in \mathbb{F}$. Then the following (i)–(iii) are equivalent:*

- (i) *For $0 \leq i, j \leq d$ such that $|i - j| = 1$,*

$$\frac{t_j}{t_i} + \frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}} = 0. \quad (2)$$

(ii) For $1 \leq i \leq d$,

$$\frac{t_i}{t_{i-1}} = -a^{-1}q^{d-2i+1}.$$

(iii) There exists $0 \neq \varepsilon \in \mathbb{F}$ such that

$$t_i = \varepsilon(-1)^i a^{-i} q^{i(d-i)} \quad (0 \leq i \leq d).$$

Proof. (i) \Leftrightarrow (ii) Use Lemma 2.3.

(ii) \Rightarrow (iii) By induction on i .

(iii) \Rightarrow (ii) Routine. □

We now consider when $q + q^{-1}$ is included among $\{\theta_i\}_{i=0}^d$.

Lemma 2.6. For $0 \leq i \leq d$,

$$\theta_i - q - q^{-1} = (a - q^{d-2i+1})(a - q^{d-2i-1})q^{2i-d}a^{-1}.$$

Proof. Use (1). □

Lemma 2.7. Assume that $\{\theta_i\}_{i=0}^d$ are mutually distinct. Then the following (i)–(iii) hold.

(i) Assume $a = q^{d+1}$. Then $\theta_0 = q + q^{-1}$.

(ii) Assume $a = q^{-d-1}$. Then $\theta_d = q + q^{-1}$.

(iii) Assume $a \neq q^{d+1}$ and $a \neq q^{-d-1}$. Then $\theta_i \neq q + q^{-1}$ for $0 \leq i \leq d$.

Proof. Use Lemmas 2.2, 2.6. □

Replacing the sequence $\{\theta_i\}_{i=0}^d$ by its inversion $\{\theta_{d-i}\}_{i=0}^d$ has the following effect.

Lemma 2.8. We have

$$\theta_{d-i} = a^{-1}q^{2i-d} + aq^{d-2i} \quad (0 \leq i \leq d).$$

Proof. Use (1). □

3 Preliminaries about linear algebra

We will be discussing algebras. An algebra is meant to be associative and have a multiplicative identity 1. A subalgebra has the same 1 as the parent algebra. Pick an integer $d \geq 0$, and let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of the \mathbb{F} -linear maps from V to V . Let $I \in \text{End}(V)$ denote the identity map. For $A \in \text{End}(V)$ let $\langle A \rangle$ denote the \mathbb{F} -subalgebra of $\text{End}(V)$ generated by A . The element A is called *diagonalizable* whenever V is spanned by the eigenspaces of A . The element A is called *multiplicity-free* whenever A is diagonalizable, and each eigenspace of A has dimension 1. Assume A is multiplicity-free, and let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of

A. For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A for V_i . Note that $\{\theta_i\}_{i=0}^d$ are mutually distinct and contained in \mathbb{F} . For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ if $j \neq i$ ($0 \leq j \leq d$). We call E_i the *primitive idempotent* of A for V_i (or θ_i). We have (i) $E_i E_j = \delta_{i,j} E_i$ ($0 \leq i, j \leq d$); (ii) $I = \sum_{i=0}^d E_i$; (iii) $A E_i = \theta_i E_i = E_i A$ ($0 \leq i \leq d$); (iv) $A = \sum_{i=0}^d \theta_i E_i$; (v) $V_i = E_i V$ ($0 \leq i \leq d$); (vi) $\text{tr}(E_i) = 1$ ($0 \leq i \leq d$), where tr means trace. Moreover

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq d). \quad (3)$$

Consider the \mathbb{F} -subalgebra $\langle A \rangle$ of $\text{End}(V)$ generated by A . Then $\{A^i\}_{i=0}^d$ is a basis for the \mathbb{F} -vector space $\langle A \rangle$, and $0 = \prod_{0 \leq i \leq d} (A - \theta_i I)$. Moreover $\{E_i\}_{i=0}^d$ is a basis for the \mathbb{F} -vector space $\langle A \rangle$. Here is another basis for the \mathbb{F} -vector space $\langle A \rangle$:

$$(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I) \quad (0 \leq i \leq d). \quad (4)$$

In this basis the primitive idempotents look as follows. By (3) and [34, Lemma 5.4] we have

$$E_i = \sum_{j=i}^d \frac{(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{j-1} I)}{(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_j)} \quad (5)$$

for $0 \leq i \leq d$. Note that for $X \in \text{End}(V)$ the following are equivalent: (i) $X \in \langle A \rangle$; (ii) $AX = XA$; (iii) $E_i X E_j = 0$ if $i \neq j$ ($0 \leq i, j \leq d$); (iv) $X = \sum_{i=0}^d E_i X E_i$. For $X \in \text{End}(V)$ we have $E_i X E_i = \text{tr}(X E_i) E_i$ ($0 \leq i \leq d$). Let $\text{Mat}_{d+1}(\mathbb{F})$ denote the \mathbb{F} -algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in \mathbb{F} . We index the rows and columns by $0, 1, \dots, d$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . For $A \in \text{End}(V)$ and $X \in \text{Mat}_{d+1}(\mathbb{F})$, we say that X *represents* A with respect to $\{v_i\}_{i=0}^d$ whenever $A v_j = \sum_{i=0}^d X_{ij} v_i$ for $0 \leq j \leq d$. A matrix $X \in \text{Mat}_{d+1}(\mathbb{F})$ is called *tridiagonal* whenever each nonzero entry is on the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is called *irreducible* whenever each entry on the subdiagonal is nonzero, and each entry on the superdiagonal is nonzero.

4 Leonard triples

In this section we recall the definition and basic facts about Leonard triples.

Definition 4.1. (See [8, Definition 1.2].) Let V denote a vector space over \mathbb{F} with finite positive dimension. By a *Leonard triple* on V we mean a 3-tuple A, B, C of elements in $\text{End}(V)$ such that

- (i) there exists a basis for V with respect to which the matrix representing A is diagonal and the matrices representing B and C are irreducible tridiagonal;
- (ii) there exists a basis for V with respect to which the matrix representing B is diagonal and the matrices representing C and A are irreducible tridiagonal;

- (iii) there exists a basis for V with respect to which the matrix representing C is diagonal and the matrices representing A and B are irreducible tridiagonal.

We say that the Leonard triple A, B, C is *over* \mathbb{F} . By the *diameter* of A, B, C we mean $d := \dim(V) - 1$.

Lemma 4.2. *Let A, B, C denote a Leonard triple on V . Then each permutation of A, B, C is a Leonard triple on V . Moreover, any two of A, B, C form a Leonard pair on V .*

Lemma 4.3. (See [38, Corollary 3.2].) *Let A, B, C denote a Leonard triple on V . Then any two of A, B, C generate the \mathbb{F} -algebra $\text{End}(V)$.*

Definition 4.4. (See [8, Definition 8.2].) Let A, B, C denote a Leonard triple on V . Let \mathcal{V} denote a vector space over \mathbb{F} with finite positive dimension, and let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ denote a Leonard triple on \mathcal{V} . An *isomorphism of Leonard triples from A, B, C to $\mathcal{A}, \mathcal{B}, \mathcal{C}$* is an \mathbb{F} -linear bijection $\sigma : V \rightarrow \mathcal{V}$ such that $\mathcal{A}\sigma = \sigma A$ and $\mathcal{B}\sigma = \sigma B$ and $\mathcal{C}\sigma = \sigma C$. The Leonard triples A, B, C and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are called *isomorphic* whenever there exists an isomorphism of Leonard triples from A, B, C to $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

For the rest of this section let A, B, C denote a Leonard triple on V , as in Definition 4.1. By [37, Lemma 1.3] each of A, B, C is multiplicity-free. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A . For $0 \leq i \leq d$ let $0 \neq v_i \in V$ denote an eigenvector of A for θ_i . Note that the sequence $\{v_i\}_{i=0}^d$ is a basis for V . The ordering $\{\theta_i\}_{i=0}^d$ is called *standard* whenever the basis $\{v_i\}_{i=0}^d$ satisfies Definition 4.1(i). Assume that the ordering $\{\theta_i\}_{i=0}^d$ is standard. Then by [28, Lemma 2.4] the ordering $\{\theta_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. Similar comments apply to B and C . For the rest of this section let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta'_i\}_{i=0}^d$) (resp. $\{\theta''_i\}_{i=0}^d$) denote a standard ordering of the eigenvalues for A (resp. B) (resp. C). Let $\{E_i\}_{i=0}^d$ (resp. $\{E'_i\}_{i=0}^d$) (resp. $\{E''_i\}_{i=0}^d$) denote the corresponding orderings of their primitive idempotents.

Lemma 4.5. (See [43, Lemma 3.3].) *For $0 \leq i, j \leq d$ the products*

$$\begin{array}{ccc} E_i B E_j, & E'_i C E'_j, & E''_i A E''_j, \\ E_i C E_j, & E'_i A E'_j, & E''_i B E''_j \end{array}$$

are zero if $|i - j| > 1$ and nonzero if $|i - j| = 1$.

Lemma 4.6. (See [37, Theorem 1.9].) *The scalars*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta'_{i-2} - \theta'_{i+1}}{\theta'_{i-1} - \theta'_i}, \quad \frac{\theta''_{i-2} - \theta''_{i+1}}{\theta''_{i-1} - \theta''_i}$$

are equal and independent of i for $2 \leq i \leq d - 1$.

In the next section we will describe a family of Leonard triples, said to have q -Racah type. Roughly speaking, this family corresponds to the “most general” solution to the constraints in Lemma 4.6.

We now formally define the modular Leonard triples. We will use the following concept. By an *antiautomorphism* of $\text{End}(V)$ we mean an \mathbb{F} -linear bijection $\dagger : \text{End}(V) \rightarrow \text{End}(V)$ such that $(XY)^\dagger = Y^\dagger X^\dagger$ for all $X, Y \in \text{End}(V)$.

Definition 4.7. (See [8, Definition 1.4].) The Leonard triple A, B, C is called *modular* whenever for each element among A, B, C there exists an antiautomorphism of $\text{End}(V)$ that fixes that element and swaps the other two elements of the triple.

5 Leonard triples of q -Racah type

In this section we describe a family of Leonard triples, said to have q -Racah type. For the rest of this paper the following notation is in effect. Fix an integer $d \geq 0$. Fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. Fix nonzero $a, b, c \in \mathbb{F}$. For the rest of this paper the following assumption is in effect.

Assumption 5.1. *We assume:*

- (i) $q^{2i} \neq 1$ for $1 \leq i \leq d$;
- (ii) none of a^2, b^2, c^2 is among $q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}$;
- (iii) none of $abc, a^{-1}bc, ab^{-1}c, abc^{-1}$ is among $q^{d-1}, q^{d-3}, \dots, q^{1-d}$.

In a moment we will describe a Leonard triple A, B, C over \mathbb{F} such that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \alpha_a I, \quad (6)$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \alpha_b I, \quad (7)$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \alpha_c I \quad (8)$$

where

$$\alpha_a = \frac{(a + a^{-1})(q^{d+1} + q^{-d-1}) + (b + b^{-1})(c + c^{-1})}{q + q^{-1}}, \quad (9)$$

$$\alpha_b = \frac{(b + b^{-1})(q^{d+1} + q^{-d-1}) + (c + c^{-1})(a + a^{-1})}{q + q^{-1}}, \quad (10)$$

$$\alpha_c = \frac{(c + c^{-1})(q^{d+1} + q^{-d-1}) + (a + a^{-1})(b + b^{-1})}{q + q^{-1}} \quad (11)$$

and d is the diameter.

Definition 5.2. For $0 \leq i \leq d$ define

$$\theta_i = aq^{2i-d} + a^{-1}q^{d-2i}, \quad \theta'_i = bq^{2i-d} + b^{-1}q^{d-2i}, \quad \theta''_i = cq^{2i-d} + c^{-1}q^{d-2i}. \quad (12)$$

Lemma 5.3. *We have*

$$\theta_i \neq \theta_j, \quad \theta'_i \neq \theta'_j, \quad \theta''_i \neq \theta''_j \quad \text{if } i \neq j, \quad (0 \leq i, j \leq d).$$

Proof. By Lemma 2.2 and Assumption 5.1. □

Definition 5.4. A Leonard triple A, B, C over \mathbb{F} is said to have *q-Racah type with Huang data* (a, b, c, d) whenever the following (i), (ii) hold:

- (i) the sequence $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta'_i\}_{i=0}^d$) (resp. $\{\theta''_i\}_{i=0}^d$) is a standard ordering of the eigenvalues for A (resp. B) (resp. C);
- (ii) A, B, C satisfy (6)–(11).

Note 5.5. The above definition of *q-Racah type* is slightly different from the one given in [26, Definition 12.1]. We make the adjustment in order to allow $d \leq 2$. For $d \geq 3$ the two definitions are equivalent by [26, Theorem 16.4].

Definition 5.6. For $1 \leq i \leq d$ define

$$\begin{aligned}\varphi_i &= a^{-1}b^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^{-i} - abcq^{i-d-1})(q^{-i} - abc^{-1}q^{i-d-1}), \\ \phi_i &= ab^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^{-i} - a^{-1}bcq^{i-d-1})(q^{-i} - a^{-1}bc^{-1}q^{i-d-1}).\end{aligned}$$

Lemma 5.7. *The scalars φ_i, ϕ_i are nonzero for $1 \leq i \leq d$.*

Proof. By Assumption 5.1. □

Proposition 5.8. (See [26, Section 16].) *There exists a Leonard triple A, B, C over \mathbb{F} that has q-Racah type and Huang data (a, b, c, d) . Up to isomorphism this Leonard triple is uniquely determined by q and (a, b, c, d) . In one basis A, B are represented by*

$$\begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & 1 & \theta_1 & & \\ & & 1 & \theta_2 & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \quad \begin{pmatrix} \theta'_0 & \varphi_1 & & & \mathbf{0} \\ & \theta'_1 & \varphi_2 & & \\ & & \theta'_2 & \cdot & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & \cdot & \varphi_d & \theta'_d \end{pmatrix}$$

respectively. In another basis A, B are represented by

$$\begin{pmatrix} \theta_d & & & & \mathbf{0} \\ & 1 & \cdot & & \\ & & \ddots & \ddots & \\ & & & \cdot & \theta_2 \\ & & & & 1 & \theta_1 \\ \mathbf{0} & & & & & 1 & \theta_0 \end{pmatrix}, \quad \begin{pmatrix} \theta'_0 & \phi_1 & & & \mathbf{0} \\ & \theta'_1 & \phi_2 & & \\ & & \theta'_2 & \cdot & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & \cdot & \phi_d & \theta'_d \end{pmatrix}$$

respectively. In either basis the matrix representing C is found using (8).

Proof. First assume that $d \geq 3$ and \mathbb{F} is algebraically closed. Then the result follows from [26, Theorem 16.4] and the discussion below [26, Definition 16.2]. One checks that the result remains valid if the two assumptions are removed. □

Note 5.9. It can happen that two distinct Huang data sequences correspond to isomorphic Leonard triples of q -Racah type. In other words, a given Leonard triple of q -Racah type can have more than one Huang data sequence. Assume the Leonard triple A, B, C has q -Racah type and Huang data (a, b, c, d) . Then by [26, Lemma 4.8], each of $(a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d)$ is a Huang data for A, B, C and A, B, C has no other Huang data.

Definition 5.10. Referring to the Leonard triple A, B, C in Definition 5.4, for $0 \leq i \leq d$ let E_i (resp. E'_i) (resp. E''_i) denote the primitive idempotent of A (resp. B) (resp. C) for θ_i (resp. θ'_i) (resp. θ''_i).

We mention some properties for Leonard triples of q -Racah type.

Lemma 5.11. *Assume the Leonard triple A, B, C has q -Racah type, with Huang data (a, b, c, d) . Then the Leonard triple B, C, A has q -Racah type, with Huang data (b, c, a, d) . Moreover the Leonard triple B, A, C has q^{-1} -Racah type, with Huang data (b, a, c, d) .*

Proof. By (6)–(11) and Definition 5.4. □

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}.$$

Lemma 5.12. *Assume the Leonard triple A, B, C has q -Racah type, with Huang data (a, b, c, d) . Then the traces of A, B, C are as follows.*

$$\text{tr}(A) = (a + a^{-1})[d + 1]_q, \quad \text{tr}(B) = (b + b^{-1})[d + 1]_q, \quad \text{tr}(C) = (c + c^{-1})[d + 1]_q.$$

Proof. Concerning A we have $\text{tr}(A) = \sum_{i=0}^d \theta_i$; evaluate this sum using the formula for θ_i in Definition 5.2. □

Lemma 5.13. *Assume that the Leonard triple A, B, C has q -Racah type, with Huang data (a, b, c, d) . Then*

$$\begin{aligned} A^2B - (q^2 + q^{-2})ABA + BA^2 + (q^2 - q^{-2})^2B &= \alpha_b(q^2 - q^{-2})^2I - \alpha_c(q - q^{-1})(q^2 - q^{-2})A, \\ B^2C - (q^2 + q^{-2})BCB + CB^2 + (q^2 - q^{-2})^2C &= \alpha_c(q^2 - q^{-2})^2I - \alpha_a(q - q^{-1})(q^2 - q^{-2})B, \\ C^2A - (q^2 + q^{-2})CAC + AC^2 + (q^2 - q^{-2})^2A &= \alpha_a(q^2 - q^{-2})^2I - \alpha_b(q - q^{-1})(q^2 - q^{-2})C \end{aligned}$$

and also

$$\begin{aligned} A^2C - (q^2 + q^{-2})ACA + CA^2 + (q^2 - q^{-2})^2C &= \alpha_c(q^2 - q^{-2})^2I - \alpha_b(q - q^{-1})(q^2 - q^{-2})A, \\ B^2A - (q^2 + q^{-2})BAB + AB^2 + (q^2 - q^{-2})^2A &= \alpha_a(q^2 - q^{-2})^2I - \alpha_c(q - q^{-1})(q^2 - q^{-2})B, \\ C^2B - (q^2 + q^{-2})CBC + BC^2 + (q^2 - q^{-2})^2B &= \alpha_b(q^2 - q^{-2})^2I - \alpha_a(q - q^{-1})(q^2 - q^{-2})C. \end{aligned}$$

Proof. To obtain the first equation, eliminate C in (7) using (8). To obtain the fourth equation, eliminate B in (8) using (7). The remaining equations are similarly obtained. □

6 Some trace formulae

Let V denote a vector space over \mathbb{F} with dimension $d+1$. For the rest of this paper A, B, C denotes a Leonard triple on V , with q -Racah type and Huang data (a, b, c, d) .

Lemma 6.1. *For $0 \leq i \leq d$ we have*

$$\begin{aligned} \left(\operatorname{tr}(BE_i) \frac{q + q^{-1} + \theta_i}{q + q^{-1}} - \alpha_c \right) \left(1 - \frac{\theta_i}{q + q^{-1}} \right) &= \alpha_b - \alpha_c, \\ \left(\operatorname{tr}(CE'_i) \frac{q + q^{-1} + \theta'_i}{q + q^{-1}} - \alpha_a \right) \left(1 - \frac{\theta'_i}{q + q^{-1}} \right) &= \alpha_c - \alpha_a, \\ \left(\operatorname{tr}(AE''_i) \frac{q + q^{-1} + \theta''_i}{q + q^{-1}} - \alpha_b \right) \left(1 - \frac{\theta''_i}{q + q^{-1}} \right) &= \alpha_a - \alpha_b \end{aligned}$$

and also

$$\begin{aligned} \left(\operatorname{tr}(CE_i) \frac{q + q^{-1} + \theta_i}{q + q^{-1}} - \alpha_b \right) \left(1 - \frac{\theta_i}{q + q^{-1}} \right) &= \alpha_c - \alpha_b, \\ \left(\operatorname{tr}(AE'_i) \frac{q + q^{-1} + \theta'_i}{q + q^{-1}} - \alpha_c \right) \left(1 - \frac{\theta'_i}{q + q^{-1}} \right) &= \alpha_a - \alpha_c, \\ \left(\operatorname{tr}(BE''_i) \frac{q + q^{-1} + \theta''_i}{q + q^{-1}} - \alpha_a \right) \left(1 - \frac{\theta''_i}{q + q^{-1}} \right) &= \alpha_b - \alpha_a. \end{aligned}$$

Proof. In the first equation of Lemma 5.13, multiply each term on the left and right by E_i , and simplify using $E_i A = \theta_i E_i = A E_i$. In the resulting equation take the trace of each side, and simplify using $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ along with $E_i^2 = E_i$ and $\operatorname{tr}(E_i) = 1$. This yields

$$\operatorname{tr}(BE_i) \frac{(q + q^{-1} - \theta_i)(q + q^{-1} + \theta_i)}{q + q^{-1}} = \alpha_b(q + q^{-1}) - \alpha_c \theta_i. \quad (13)$$

Rearranging the terms in (13) we obtain the first equation of the present lemma. The remaining equations of the present lemma are similarly obtained. \square

Lemma 6.2. *We have*

$$\alpha_a - \alpha_b = \frac{(b - a)(b - a^{-1})(c - q^{d+1})(c - q^{-d-1})b^{-1}c^{-1}}{q + q^{-1}}, \quad (14)$$

$$\alpha_b - \alpha_c = \frac{(c - b)(c - b^{-1})(a - q^{d+1})(a - q^{-d-1})c^{-1}a^{-1}}{q + q^{-1}}, \quad (15)$$

$$\alpha_c - \alpha_a = \frac{(a - c)(a - c^{-1})(b - q^{d+1})(b - q^{-d-1})a^{-1}b^{-1}}{q + q^{-1}}. \quad (16)$$

Proof. Use (9)–(11). \square

7 The element W

We continue to discuss the Leonard triple A, B, C on V with q -Racah type and Huang data (a, b, c, d) . Our next general goal is to find all the invertible $W \in \text{End}(V)$ such that A commutes with W and $W^{-1}BW - C$. We first record some results from linear algebra.

Lemma 7.1. *For $W \in \text{End}(V)$ the following are equivalent:*

- (i) W commutes with A ;
- (ii) there exist scalars $\{t_i\}_{i=0}^d$ in \mathbb{F} such that

$$W = \sum_{i=0}^d t_i E_i. \quad (17)$$

Lemma 7.2. *Assume $W \in \text{End}(V)$ satisfies the equivalent conditions (i), (ii) in Lemma 7.1. Then the following are equivalent:*

- (i) W is invertible;
- (ii) $t_i \neq 0$ for $0 \leq i \leq d$.

Suppose (i), (ii). Then

$$W^{-1} = \sum_{i=0}^d t_i^{-1} E_i. \quad (18)$$

Assume that $W \in \text{End}(V)$ is invertible and commutes with A . Consider the scalars $\{t_i\}_{i=0}^d$ from Lemma 7.1. We now find necessary and sufficient conditions on the $\{t_i\}_{i=0}^d$ for A to commute with $W^{-1}BW - C$.

Lemma 7.3. *Assume that $W \in \text{End}(V)$ is invertible and commutes with A . For $0 \leq i, j \leq d$ consider*

$$E_i(W^{-1}BW - C)E_j. \quad (19)$$

- (i) Assume $|i - j| > 1$. Then (19) is equal to 0.
- (ii) Assume $|i - j| = 1$. Then (19) is equal to $E_i B E_j$ times

$$\frac{t_j}{t_i} + \frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}},$$

where t_0, \dots, t_d are from Lemma 7.1.

- (iii) Assume $i = j$. Then (19) is equal to E_i times

$$\text{tr}(B E_i) \frac{q + q^{-1} + \theta_i}{q + q^{-1}} - \alpha_c.$$

Proof. In the expression (19), eliminate C using (8) and eliminate W, W^{-1} using (17), (18). Simplify the result using Lemma 4.5. \square

Proposition 7.4. *Assume that $W \in \text{End}(V)$ is invertible and commutes with A . Then the following are equivalent:*

- (i) A commutes with $W^{-1}BW - C$;
- (ii) the scalars $\{t_i\}_{i=0}^d$ from Lemma 7.1 satisfy the equivalent conditions (i)–(iii) in Lemma 2.5.

Proof. (i) \Rightarrow (ii) We show that the scalars $\{t_i\}_{i=0}^d$ satisfy Lemma 2.5(i). For $0 \leq i, j \leq d$ such that $|i - j| = 1$, let α_{ij} denote the expression on the left in (2). We show $\alpha_{ij} = 0$. By assumption (i) and Lemma 7.3(ii) we have

$$0 = E_i(W^{-1}BW - C)E_j = E_iBE_j\alpha_{ij}.$$

But $E_iBE_j \neq 0$ by Lemma 4.5, so $\alpha_{ij} = 0$.

(ii) \Rightarrow (i) By Lemma 2.5(i) and Lemma 7.3(i),(ii) we find that the expression (19) is zero for all $0 \leq i, j \leq d$ such that $i \neq j$. Therefore A commutes with $W^{-1}BW - C$. \square

Proposition 7.5. *For $W \in \text{End}(V)$ the following are equivalent:*

- (i) W is invertible, and A commutes with W and $W^{-1}BW - C$;
- (ii) there exists $0 \neq \varepsilon \in \mathbb{F}$ such that

$$W = \varepsilon \sum_{i=0}^d (-1)^i a^{-i} q^{i(d-i)} E_i.$$

Proof. By Lemma 2.5(iii) and Proposition 7.4. \square

Theorem 7.6. *There exists an invertible $W \in \text{End}(V)$ such that A commutes with W and $W^{-1}BW - C$. This W is unique up to multiplication by a nonzero scalar in \mathbb{F} .*

Proof. By Proposition 7.5. \square

8 The elements $\overline{A}, \overline{B}, \overline{C}$

We continue to discuss the Leonard triple A, B, C on V with q -Racah type and Huang data (a, b, c, d) . For the rest of this paper we adopt the following notation.

Definition 8.1. Define

$$\begin{aligned} W &= \sum_{i=0}^d (-1)^i a^{-i} q^{i(d-i)} E_i, \\ W' &= \sum_{i=0}^d (-1)^i b^{-i} q^{i(d-i)} E'_i, \\ W'' &= \sum_{i=0}^d (-1)^i c^{-i} q^{i(d-i)} E''_i. \end{aligned}$$

We emphasize a few points.

Lemma 8.2. *Each of W , W' , W'' is invertible. Moreover*

$$\begin{aligned} W^{-1} &= \sum_{i=0}^d (-1)^i a^i q^{-i(d-i)} E_i, \\ (W')^{-1} &= \sum_{i=0}^d (-1)^i b^i q^{-i(d-i)} E'_i, \\ (W'')^{-1} &= \sum_{i=0}^d (-1)^i c^i q^{-i(d-i)} E''_i. \end{aligned}$$

Proof. Use Definition 8.1. □

Lemma 8.3. *The following (i)–(iii) hold.*

- (i) A commutes with W and $W^{-1}BW - C$;
- (ii) B commutes with W' and $(W')^{-1}CW' - A$;
- (iii) C commutes with W'' and $(W'')^{-1}AW'' - B$.

Proof. Part (i) is by Proposition 7.5. Parts (ii), (iii) are similarly obtained. □

Definition 8.4. Define

$$\overline{A} = W^{-1}BW - C, \quad \overline{B} = (W')^{-1}CW' - A, \quad \overline{C} = (W'')^{-1}AW'' - B.$$

Lemma 8.5. *We have*

$$\overline{A} \in \langle A \rangle, \quad \overline{B} \in \langle B \rangle, \quad \overline{C} \in \langle C \rangle.$$

Proof. The element A is multiplicity-free, and \overline{A} commutes with A , so $\overline{A} \in \langle A \rangle$. The remaining assertions are similarly obtained. □

Lemma 8.6. *We have*

$$\overline{A} = B - WCW^{-1}, \quad \overline{B} = C - W'A(W')^{-1}, \quad \overline{C} = A - W''B(W'')^{-1}.$$

Proof. \overline{A} commutes with A , and $W \in \langle A \rangle$, so \overline{A} commutes with W . Therefore $\overline{A} = W\overline{A}W^{-1} = W(W^{-1}BW - C)W^{-1} = B - WCW^{-1}$. The remaining assertions are similarly obtained. □

Lemma 8.7. *We have*

$$\mathrm{tr}(\overline{A}) = \mathrm{tr}(B) - \mathrm{tr}(C), \quad \mathrm{tr}(\overline{B}) = \mathrm{tr}(C) - \mathrm{tr}(A), \quad \mathrm{tr}(\overline{C}) = \mathrm{tr}(A) - \mathrm{tr}(B).$$

Proof. Use Definition 8.4. □

Corollary 8.8. *We have*

$$\begin{aligned}\mathrm{tr}(\overline{A}) &= (b - c)(b - c^{-1})b^{-1}[d + 1]_q, \\ \mathrm{tr}(\overline{B}) &= (c - a)(c - a^{-1})c^{-1}[d + 1]_q, \\ \mathrm{tr}(\overline{C}) &= (a - b)(a - b^{-1})a^{-1}[d + 1]_q.\end{aligned}$$

Proof. Use Lemmas 5.12, 8.7. □

Proposition 8.9. *We have*

$$\begin{aligned}\overline{A}\left(I - \frac{A}{q + q^{-1}}\right) &= (\alpha_b - \alpha_c)I = \left(I - \frac{A}{q + q^{-1}}\right)\overline{A}; \\ \overline{B}\left(I - \frac{B}{q + q^{-1}}\right) &= (\alpha_c - \alpha_a)I = \left(I - \frac{B}{q + q^{-1}}\right)\overline{B}; \\ \overline{C}\left(I - \frac{C}{q + q^{-1}}\right) &= (\alpha_a - \alpha_b)I = \left(I - \frac{C}{q + q^{-1}}\right)\overline{C}.\end{aligned}$$

Proof. We verify the top equations in the proposition statement. Since A, \overline{A} commute it suffices to show

$$\overline{A}\left(I - \frac{A}{q + q^{-1}}\right) = (\alpha_b - \alpha_c)I. \quad (20)$$

Let Δ denote the left-hand side of (20) minus the right-hand side of (20). We show $\Delta = 0$. We have $\Delta \in \langle A \rangle$ by Lemma 8.5, so $\Delta = \sum_{i=0}^d E_i \Delta E_i$. We show $E_i \Delta E_i = 0$ for $0 \leq i \leq d$. Let i be given. We have

$$\begin{aligned}E_i \Delta E_i &= E_i \overline{A} E_i \left(1 - \frac{\theta_i}{q + q^{-1}}\right) - (\alpha_b - \alpha_c) E_i \\ &= E_i (W^{-1} B W - C) E_i \left(1 - \frac{\theta_i}{q + q^{-1}}\right) - (\alpha_b - \alpha_c) E_i.\end{aligned} \quad (21)$$

Evaluating (21) using Lemma 7.3(iii) and the first equation in Lemma 6.1, we routinely obtain $E_i \Delta E_i = 0$. We have shown (20), and we verified the top equations in the proposition statement. The remaining equations are similarly verified. □

Consider the top equations in Proposition 8.9. As we seek to describe \overline{A} , it is tempting to invert the element $I - A/(q + q^{-1})$. However this element might not be invertible. We now investigate this possibility.

Lemma 8.10. *The following are equivalent:*

- (i) $I - \frac{A}{q + q^{-1}}$ is invertible;
- (ii) $a \neq q^{d+1}$ and $a \neq q^{-d-1}$.

Proof. Use Lemma 2.7. □

We now describe \overline{A} . Similar descriptions apply to \overline{B} and \overline{C} .

Proposition 8.11. *The element \overline{A} is described as follows.*

(i) Assume $a = q^{d+1}$. Then

$$\overline{A} = (b - c)(b - c^{-1})b^{-1}[d + 1]_q E_0. \quad (22)$$

(ii) Assume $a = q^{-d-1}$. Then

$$\overline{A} = (b - c)(b - c^{-1})b^{-1}[d + 1]_q E_d. \quad (23)$$

(iii) Assume $a \neq q^{d+1}$ and $a \neq q^{-d-1}$. Then

$$\overline{A} = (\alpha_b - \alpha_c) \left(I - \frac{A}{q + q^{-1}} \right)^{-1}. \quad (24)$$

Proof. (i) Recall that $\{E_i\}_{i=0}^d$ is a basis for $\langle A \rangle$, and $\overline{A} \in \langle A \rangle$ by Lemma 8.5. So there exist scalars $\{s_i\}_{i=0}^d$ in \mathbb{F} such that $\overline{A} = \sum_{i=0}^d s_i E_i$. By Lemma 6.2 we have $\alpha_b = \alpha_c$, so the first equation in Proposition 8.9 becomes

$$\overline{A} \left(1 - \frac{A}{q + q^{-1}} \right) = 0. \quad (25)$$

From (25) we obtain

$$s_i \left(1 - \frac{\theta_i}{q + q^{-1}} \right) = 0 \quad (0 \leq i \leq d).$$

In the above equation, for $1 \leq i \leq d$ the coefficient of s_i is nonzero by Lemma 2.7(i) and $\theta_i \neq \theta_0$. Consequently $s_i = 0$. By these comments $\overline{A} = s_0 E_0$. In this equation, take the trace of each side and use $\text{tr}(E_0) = 1$ to obtain $s_0 = \text{tr}(\overline{A})$. This and Corollary 8.8 imply (22).

(ii) Similar to the proof of (i) above.

(iii) By the first equation in Proposition 8.9, along with Lemma 8.10. \square

Here is another view of \overline{A} .

Proposition 8.12. *The element \overline{A} is described as follows.*

(i) Assume $a = q^{d+1}$. Then \overline{A} is equal to

$$(b - c)(b - c^{-1})b^{-1}[d + 1]_q \quad (26)$$

times

$$\sum_{i=0}^d \frac{(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_i)}. \quad (27)$$

(ii) Assume $a = q^{-d-1}$. Then \overline{A} is equal to

$$(b - c)(b - c^{-1})b^{-1}[d + 1]_q \quad (28)$$

times

$$\frac{(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{d-1} I)}{(\theta_d - \theta_0)(\theta_d - \theta_1) \cdots (\theta_d - \theta_{d-1})}. \quad (29)$$

(iii) Assume $a \neq q^{d+1}$ and $a \neq q^{-d-1}$. Then \overline{A} is equal to

$$\frac{(a - q^{-d-1})(b - c)(b - c^{-1})b^{-1}q^d}{a - q^{d-1}} \quad (30)$$

times

$$\sum_{i=0}^d \frac{(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q + q^{-1} - \theta_1)(q + q^{-1} - \theta_2) \cdots (q + q^{-1} - \theta_i)}. \quad (31)$$

Proof. (i) In line (22) evaluate E_0 using (5).

(ii) In line (23) evaluate E_d using (5).

(iii) Recall the basis (4) for $\langle A \rangle$. Since $\overline{A} \in \langle A \rangle$, there exist scalars $\{\gamma_i\}_{i=0}^d$ in \mathbb{F} such that

$$\overline{A} = \sum_{i=0}^d \gamma_i (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I).$$

In this equation multiply each side by $I - A/(q + q^{-1})$. Evaluate the resulting equation using Proposition 8.9, to obtain

$$\alpha_b - \alpha_c = \gamma_0 \left(1 - \frac{\theta_0}{q + q^{-1}} \right) \quad (32)$$

and $\gamma_{i-1} = \gamma_i(q + q^{-1} - \theta_i)$ for $1 \leq i \leq d$. Evaluating (32) using Lemma 2.6 and (15), we find that γ_0 is equal to (30). The result follows. \square

9 Some results about conjugation

We continue to discuss the Leonard triple A, B, C on V with q -Racah type and Huang data (a, b, c, d) . Recall the maps W, W', W'' from Definition 8.1. In this section we work out what happens if one of A, B, C is conjugated by one of $W^{\pm 1}, (W')^{\pm 1}, (W'')^{\pm 1}$. We start with some observations about W ; similar observations apply to W' and W'' .

Lemma 9.1. *We have*

$$WBW^{-1} - W^{-1}BW = \frac{AB - BA}{q - q^{-1}}, \quad (33)$$

$$WCW^{-1} - W^{-1}CW = \frac{AC - CA}{q - q^{-1}}. \quad (34)$$

Proof. Concerning (33), we have

$$\begin{aligned} W^{-1}BW &= \bar{A} + C \\ &= \bar{A} + \alpha_c I - \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \end{aligned} \quad (35)$$

and also

$$\begin{aligned} WBW^{-1} &= W \left(\alpha_b I - \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} \right) W^{-1} \\ &= \alpha_b I - \frac{qWCW^{-1}A - q^{-1}AWCW^{-1}}{q^2 - q^{-2}} \\ &= \alpha_b I - \frac{q(B - \bar{A})A - q^{-1}A(B - \bar{A})}{q^2 - q^{-2}} \\ &= \alpha_b I - \frac{qBA - q^{-1}AB}{q^2 - q^{-2}} + \frac{A\bar{A}}{q + q^{-1}}. \end{aligned} \quad (36)$$

Now subtract (35) from (36), and evaluate the result using the first equation in Proposition 8.9. This yields (33). Concerning (34), we have

$$\begin{aligned} WCW^{-1} &= B - \bar{A} \\ &= \alpha_b I - \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} - \bar{A} \end{aligned} \quad (37)$$

and also

$$\begin{aligned} W^{-1}CW &= W^{-1} \left(\alpha_c I - \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \right) W \\ &= \alpha_c I - \frac{qAW^{-1}BW - q^{-1}W^{-1}BWA}{q^2 - q^{-2}} \\ &= \alpha_c I - \frac{qA(\bar{A} + C) - q^{-1}(\bar{A} + C)A}{q^2 - q^{-2}} \\ &= \alpha_c I - \frac{qAC - q^{-1}CA}{q^2 - q^{-2}} - \frac{A\bar{A}}{q + q^{-1}}. \end{aligned} \quad (38)$$

Now subtract (38) from (37), and evaluate the result using the first equation in Proposition 8.9. This yields (34). \square

Let X denote one of A, B, C . In the next result we conjugate X by each of $W^{\pm 1}, (W')^{\pm 1}, (W'')^{\pm 1}$.

Proposition 9.2. *We have*

X	A	B	C
$W^{-1}XW$	A	$C + \bar{A}$	$B + \frac{CA - AC}{q - q^{-1}} - \bar{A}$
WXW^{-1}	A	$C + \frac{AB - BA}{q - q^{-1}} + \bar{A}$	$B - \bar{A}$
$(W')^{-1}XW'$	$C + \frac{AB - BA}{q - q^{-1}} - \bar{B}$	B	$A + \bar{B}$
$W'X(W')^{-1}$	$C - \bar{B}$	B	$A + \frac{BC - CB}{q - q^{-1}} + \bar{B}$
$(W'')^{-1}XW''$	$B + \bar{C}$	$A + \frac{BC - CB}{q - q^{-1}} - \bar{C}$	C
$W''X(W'')^{-1}$	$B + \frac{CA - AC}{q - q^{-1}} + \bar{C}$	$A - \bar{C}$	C

Proof. Use Definition 8.4 and Lemmas 8.5, 8.6, 9.1. □

10 More results about conjugation

We continue to discuss the Leonard triple A, B, C on V with q -Racah type and Huang data (a, b, c, d) . Recall the maps W, W', W'' from Definition 8.1. In this section we work out what happens if one of A, B, C is conjugated by one of $W^{\pm 2}, (W')^{\pm 2}, (W'')^{\pm 2}$.

Lemma 10.1. *We have*

$$W^2 = \sum_{i=0}^d a^{-2i} q^{2i(d-i)} E_i, \quad W^{-2} = \sum_{i=0}^d a^{2i} q^{-2i(d-i)} E_i, \quad (39)$$

$$(W')^2 = \sum_{i=0}^d b^{-2i} q^{2i(d-i)} E'_i, \quad (W')^{-2} = \sum_{i=0}^d b^{2i} q^{-2i(d-i)} E'_i, \quad (40)$$

$$(W'')^2 = \sum_{i=0}^d c^{-2i} q^{2i(d-i)} E''_i, \quad (W'')^{-2} = \sum_{i=0}^d c^{2i} q^{-2i(d-i)} E''_i. \quad (41)$$

Proof. Use Definition 8.1. □

We mention a result about W^2 ; similar results apply to $(W')^2$ and $(W'')^2$. Recall the commutator $[X, Y] = XY - YX$.

Lemma 10.2. *We have*

$$W^2 B W^{-2} + W^{-2} B W^2 = 2B + \frac{[A, [A, B]]}{(q - q^{-1})^2}, \quad (42)$$

$$W^2 C W^{-2} + W^{-2} C W^2 = 2C + \frac{[A, [A, C]]}{(q - q^{-1})^2}. \quad (43)$$

Proof. Concerning (42), observe

$$\begin{aligned} & W^2 B W^{-2} + W^{-2} B W^2 - 2B \\ &= W(W B W^{-1} - W^{-1} B W) W^{-1} - W^{-1}(W B W^{-1} - W^{-1} B W) W \\ &= \frac{W[A, B] W^{-1} - W^{-1}[A, B] W}{q - q^{-1}} \\ &= \frac{[A, W B W^{-1} - W^{-1} B W]}{q - q^{-1}} \\ &= \frac{[A, [A, B]]}{(q - q^{-1})^2}. \end{aligned}$$

The proof of (43) is similar. □

Let X denote one of A, B, C . In the next result we conjugate X by each of $W^{\pm 2}, (W')^{\pm 2}, (W'')^{\pm 2}$.

Proposition 10.3. *We have*

X	A	B	C
$W^{-2}XW^2$	A	$B + \frac{[C,A]}{q-q^{-1}}$	$C - \frac{[A,B]}{q-q^{-1}} + \frac{[A,[A,C]]}{(q-q^{-1})^2}$
W^2XW^{-2}	A	$B - \frac{[C,A]}{q-q^{-1}} + \frac{[A,[A,B]]}{(q-q^{-1})^2}$	$C + \frac{[A,B]}{q-q^{-1}}$
$(W')^{-2}X(W')^2$	$A - \frac{[B,C]}{q-q^{-1}} + \frac{[B,[B,A]]}{(q-q^{-1})^2}$	B	$C + \frac{[A,B]}{q-q^{-1}}$
$(W')^2X(W')^{-2}$	$A + \frac{[B,C]}{q-q^{-1}}$	B	$C - \frac{[A,B]}{q-q^{-1}} + \frac{[B,[B,C]]}{(q-q^{-1})^2}$
$(W'')^{-2}X(W'')^2$	$A + \frac{[B,C]}{q-q^{-1}}$	$B - \frac{[C,A]}{q-q^{-1}} + \frac{[C,[C,B]]}{(q-q^{-1})^2}$	C
$(W'')^2X(W'')^{-2}$	$A - \frac{[B,C]}{q-q^{-1}} + \frac{[C,[C,A]]}{(q-q^{-1})^2}$	$B + \frac{[C,A]}{q-q^{-1}}$	C

Proof. Consider the rows of the above table that involve W . To verify these rows, repeatedly apply Proposition 9.2 and use the fact that W commutes with \overline{A} . Lemma 10.2 can be used to simplify the calculation. The rows of the table involving W' , W'' are similarly verified. \square

We mention an alternate way to express the results in Proposition 10.3. We focus on the part involving W .

Lemma 10.4. *We have*

$$W^{-2}BW^2 = B + \frac{qA^2B - (q + q^{-1})ABA + q^{-1}BA^2}{(q - q^{-1})(q^2 - q^{-2})},$$

$$W^2BW^{-2} = B + \frac{q^{-1}A^2B - (q + q^{-1})ABA + qBA^2}{(q - q^{-1})(q^2 - q^{-2})}.$$

Proof. In the formula for $W^{-2}BW^2$ and W^2BW^{-2} given in Proposition 10.3, eliminate C using (8). \square

Lemma 10.5. *We have*

$$W^{-2}CW^2 = C + \frac{qA^2C - (q + q^{-1})ACA + q^{-1}CA^2}{(q - q^{-1})(q^2 - q^{-2})},$$

$$W^2CW^{-2} = C + \frac{q^{-1}A^2C - (q + q^{-1})ACA + qCA^2}{(q - q^{-1})(q^2 - q^{-2})}.$$

Proof. In the formula for $W^{-2}CW^2$ and W^2CW^{-2} given in Proposition 10.3, eliminate B using (7). \square

Note 10.6. Lemmas 10.4, 10.5 indicate how W^2 , $(W')^2$, $(W'')^2$ are related to the Lusztig automorphisms considered by Baseilhac and Kolb [4].

11 The elements $W^{\pm 1}$, $W^{\pm 2}$ as a polynomial in A

We continue to discuss the Leonard triple A, B, C on V with q -Racah type and Huang data (a, b, c, d) . Recall the map W from Definition 8.1. In this section we express each of $W^{\pm 1}$, $W^{\pm 2}$ as a polynomial in A . First we recall some notation. For $x, t \in \mathbb{F}$,

$$(x; t)_n = (1 - x)(1 - xt) \cdots (1 - xt^{n-1}) \quad n = 0, 1, 2, \dots$$

We interpret $(x; t)_0 = 1$.

Proposition 11.1. *We have*

$$W = \sum_{i=0}^d \frac{(-1)^i q^{i^2} (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q^2; q^2)_i (aq^{1-d}; q^2)_i}, \quad (44)$$

$$W^{-1} = \sum_{i=0}^d \frac{(-1)^i a^i q^{i(i-d+1)} (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q^2; q^2)_i (aq^{1-d}; q^2)_i}. \quad (45)$$

Proof. It is convenient to prove (45) first. Each side of (45) is a polynomial in A . For $0 \leq j \leq d$ we show that the two sides have the same eigenvalue for E_j . This amounts to showing

$$(-1)^j a^j q^{j(j-d)} = \sum_{i=0}^d \frac{(-1)^i a^i q^{i(i-d+1)} (\theta_j - \theta_0)(\theta_j - \theta_1) \cdots (\theta_j - \theta_{i-1})}{(q^2; q^2)_i (aq^{1-d}; q^2)_i}. \quad (46)$$

Evaluating (46) using Lemma 2.1, we find it becomes the following special case of the q^2 -Chu-Vandermonde identity [13, p. 236]:

$$(-1)^j a^j q^{j(j-d)} = {}_2\phi_1 \left(\begin{matrix} q^{-2j}, a^2 q^{2j-2d} \\ aq^{1-d} \end{matrix} \middle| q^2, q^2 \right). \quad (47)$$

We are using the basic hypergeometric series notation [13, p. 3]. For (44) the proof is similar. For $0 \leq j \leq d$ we show that the two sides have the same eigenvalue for E_j . This amounts to showing

$$(-1)^j a^{-j} q^{j(d-j)} = \sum_{i=0}^d \frac{(-1)^i q^{i^2} (\theta_j - \theta_0)(\theta_j - \theta_1) \cdots (\theta_j - \theta_{i-1})}{(q^2; q^2)_i (aq^{1-d}; q^2)_i}. \quad (48)$$

Evaluating (48) using Lemma 2.1, we find it becomes the following identity, which is just (47) with a, q replaced by a^{-1}, q^{-1} .

$$(-1)^j a^{-j} q^{j(d-j)} = {}_2\phi_1 \left(\begin{matrix} q^{2j}, a^{-2} q^{2d-2j} \\ a^{-1} q^{d-1} \end{matrix} \middle| q^{-2}, q^{-2} \right).$$

□

Proposition 11.2. *We have*

$$W^2 = \sum_{i=0}^d \frac{a^{-i} q^{id} (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q^2; q^2)_i}, \quad (49)$$

$$W^{-2} = \sum_{i=0}^d \frac{(-1)^i a^i q^{i(i-d+1)} (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)}{(q^2; q^2)_i}. \quad (50)$$

Proof. Similar to the proof of Proposition 11.1. First consider (50). For $0 \leq j \leq d$ we show that the two sides of (50) have the same eigenvalue for E_j . This amounts to showing

$$a^{2j} q^{2j(j-d)} = \sum_{i=0}^d \frac{(-1)^i a^i q^{i(i-d+1)} (\theta_j - \theta_0)(\theta_j - \theta_1) \cdots (\theta_j - \theta_{i-1})}{(q^2; q^2)_i}. \quad (51)$$

Evaluating (51) using Lemma 2.1, we find it reduces to the following special case of the q^2 -Chu-Vandermonde identity [13, p. 236]:

$$a^{2j} q^{2j(j-d)} = {}_2\phi_1 \left(\begin{matrix} q^{-2j}, a^2 q^{2j-2d} \\ 0 \end{matrix} \middle| q^2, q^2 \right). \quad (52)$$

Next consider (49). For $0 \leq j \leq d$ we show that the two sides of (49) have the same eigenvalue for E_j . This amounts to showing

$$a^{-2j} q^{2j(d-j)} = \sum_{i=0}^d \frac{a^{-i} q^{id} (\theta_j - \theta_0)(\theta_j - \theta_1) \cdots (\theta_j - \theta_{i-1})}{(q^2; q^2)_i}. \quad (53)$$

Evaluating (53) using Lemma 2.1, we find it reduces to the following identity, which is just (52) with a, q replaced by a^{-1}, q^{-1} .

$$a^{-2j} q^{2j(d-j)} = {}_2\phi_1 \left(\begin{matrix} q^{2j}, a^{-2} q^{2d-2j} \\ 0 \end{matrix} \middle| q^{-2}, q^{-2} \right).$$

□

12 A product

We continue to discuss the Leonard triple A, B, C on V with q -Racah type and Huang data (a, b, c, d) . Recall the maps W, W', W'' from Definition 8.1. In this section we compute the product $(W'')^2(W')^2W^2$.

Proposition 12.1. *We have*

$$(W'')^2(W')^2W^2 = (abc)^{-d} q^{d(d-1)} I.$$

Proof. We first show that $(W'')^2(W')^2W^2$ commutes with A . Since W commutes with A , it suffices to show that $(W'')^2(W')^2$ commutes with A . From Proposition 10.3 we obtain $(W')^2 A (W')^{-2} = (W'')^{-2} A (W'')^2$. Therefore $(W'')^2(W')^2$ commutes with A . Next we show that $(W'')^2(W')^2W^2$ commutes with C . Since W'' commutes with C , it suffices to show that $(W')^2W^2$ commutes with C . From Proposition 10.3 we obtain $W^2 C W^{-2} = (W')^{-2} C (W')^2$. Therefore $(W')^2W^2$ commutes with C . We have shown that $(W'')^2(W')^2W^2$ commutes with A and C . By Lemma 4.3 the elements A, C generate $\text{End}(V)$. Therefore $(W'')^2(W')^2W^2$ is central in $\text{End}(V)$. Consequently there exists $\theta \in \mathbb{F}$ such that $(W'')^2(W')^2W^2 = \theta I$. We now compute θ . For the rest of this proof, we identify A and B with their matrix

representations from the first display Proposition 5.8. The matrix C is obtained using (8). The matrices A, B, C are lower bidiagonal, upper bidiagonal, and tridiagonal, respectively. By (39) and the construction, the matrix W^2 is lower triangular with (i, i) -entry $a^{-2i}q^{2i(d-i)}$ for $0 \leq i \leq d$. By (40) and the construction, the matrix $(W')^2$ is upper triangular with (i, i) -entry $b^{-2i}q^{2i(d-i)}$ for $0 \leq i \leq d$. Note that $(W')^2W^2 = \theta(W'')^{-2}$. In this equation we compute the $(d, 0)$ -entry of each side. For the left-hand side, the $(d, 0)$ -entry is equal to the (d, d) -entry of $(W')^2$ times the $(d, 0)$ -entry of W^2 . The (d, d) -entry of $(W')^2$ is b^{-2d} . Next we compute the $(d, 0)$ -entry of W^2 . By (49) the element W^2 is a polynomial in A with degree d and A^d -coefficient $a^{-d}q^{d^2}/(q^2; q^2)_d$. The matrix A is lower bidiagonal with $(i, i-1)$ -entry 1 for $1 \leq i \leq d$. Therefore the $(d, 0)$ -entry of A^d is 1, and the $(d, 0)$ -entry of A^j is 0 for $0 \leq j \leq d-1$. By these comments the $(d, 0)$ -entry of W^2 is equal to $a^{-d}q^{d^2}/(q^2; q^2)_d$. Next we compute the $(d, 0)$ -entry of $(W'')^{-2}$. Applying (50) to W'' , we find that $(W'')^{-2}$ is a polynomial in C with degree d and C^d -coefficient $(-1)^d c^d q^d / (q^2; q^2)_d$. We mentioned that C is tridiagonal; for $1 \leq i \leq d$ we compute its $(i, i-1)$ -entry. Using (8) and the form of A, B we find that C has $(i, i-1)$ -entry $(q^{-1}\theta'_i - q\theta'_{i-1})(q^2 - q^{-2})^{-1}$ which is equal to $-b^{-1}q^{d-2i+1}$. Denote this quantity by β_i . The $(d, 0)$ -entry of C^d is $\beta_1\beta_2 \cdots \beta_d$ which is equal to $(-1)^d b^{-d}$. The $(d, 0)$ -entry of C^j is 0 for $0 \leq j \leq d-1$. By these comments the $(d, 0)$ -entry of $(W'')^{-2}$ is $b^{-d}c^d q^d / (q^2; q^2)_d$. Putting it all together we obtain

$$\frac{b^{-2d}a^{-d}q^{d^2}}{(q^2; q^2)_d} = \frac{\theta b^{-d}c^d q^d}{(q^2; q^2)_d},$$

so $\theta = (abc)^{-d}q^{d(d-1)}$. The result follows. \square

13 The elements $W'W$, $W''W'$, WW''

We continue to discuss the Leonard triple A, B, C on V with q -Racah type and Huang data (a, b, c, d) . Recall the maps W, W', W'' from Definition 8.1. In this section we consider the three elements

$$W'W, \quad W''W', \quad WW''. \quad (54)$$

We show that these elements mutually commute, and their product is $(abc)^{-d}q^{d(d-1)}I$.

Lemma 13.1. *Each of the elements (54) commutes with $A + B + C$.*

Proof. We show that $W'W$ commutes with $A + B + C$. Using Proposition 9.2 we obtain

$$W(A + B + C)W^{-1} = A + B + C + \frac{AB - BA}{q - q^{-1}} = (W')^{-1}(A + B + C)W'.$$

It follows that $W'W$ commutes with $A + B + C$. \square

Recall that $\langle A + B + C \rangle$ is the \mathbb{F} -subalgebra of $\text{End}(V)$ generated by $A + B + C$.

Lemma 13.2. *There exists $v \in V$ such that $\langle A + B + C \rangle v = V$.*

Proof. Let $\{v_i\}_{i=0}^d$ denote the first basis for V referred to in Proposition 5.8. We show that $v = v_0$ satisfies the requirements of the present lemma. With respect to the basis $\{v_i\}_{i=0}^d$ the matrices representing A and B are lower bidiagonal and upper bidiagonal, respectively. Moreover by (8) the matrix representing C is tridiagonal. Let $X \in \text{Mat}_{d+1}(\mathbb{F})$ represent $A+B+C$ with respect to $\{v_i\}_{i=0}^d$. The matrix X is tridiagonal; we show that X is irreducible. Using (8) we find that for $1 \leq i \leq d$ the $(i, i-1)$ -entry of X is $1 - b^{-1}q^{d-2i+1}$, which is nonzero by Assumption 5.1(ii). Similarly the $(i-1, i)$ -entry of X is $\varphi_i(1 - a^{-1}q^{d-2i+1})$, which is nonzero by Assumption 5.1(ii) and Lemma 5.7. By these comments X is irreducible. It follows that for $0 \leq i \leq d$ there exists a polynomial f_i with coefficients in \mathbb{F} and degree i such that $f_i(A+B+C)v_0 = v_i$. The vectors $\{v_i\}_{i=0}^d$ span V so $\langle A+B+C \rangle v_0 = V$. \square

Corollary 13.3. *The subalgebra $\langle A+B+C \rangle$ contains every element of $\text{End}(V)$ that commutes with $A+B+C$. In particular $\langle A+B+C \rangle$ contains each of the elements (54).*

Proof. Assume $G \in \text{End}(V)$ commutes with $A+B+C$. We show that $G \in \langle A+B+C \rangle$. Recall the vector v from Lemma 13.2, and consider Gv . By Lemma 13.2 there exists $H \in \langle A+B+C \rangle$ such that $Gv = Hv$. Note that $G-H$ commutes with $A+B+C$. We may now argue

$$\begin{aligned} (G-H)V &= (G-H)\langle A+B+C \rangle v \\ &= \langle A+B+C \rangle (G-H)v \\ &= \langle A+B+C \rangle 0 \\ &= 0. \end{aligned}$$

Therefore $G = H \in \langle A+B+C \rangle$. \square

Theorem 13.4. *The elements in (54) mutually commute.*

Proof. By Corollary 13.3 and since the algebra $\langle A+B+C \rangle$ is commutative. \square

Theorem 13.5. *The product of the elements (54) is equal to $(abc)^{-d}q^{d(d-1)}I$.*

Proof. Using Proposition 12.1 and Theorem 13.4 we obtain

$$\begin{aligned} (W'W)(W''W')(WW'') &= (WW'')(W''W')(W'W) \\ &= W(W'')^2(W')^2W^2W^{-1} \\ &= (abc)^{-d}q^{d(d-1)}WW^{-1} \\ &= (abc)^{-d}q^{d(d-1)}I. \end{aligned}$$

\square

14 The case $\overline{A} = \overline{B} = \overline{C} = 0$

We continue to discuss the Leonard triple A, B, C on V with q -Racah type and Huang data (a, b, c, d) . Recall the maps $\overline{A}, \overline{B}, \overline{C}$ from Definition 8.4. In this section we consider the case in which $\overline{A} = \overline{B} = \overline{C} = 0$.

Definition 14.1. We define a binary relation \sim on \mathbb{F} called *similarity*. For $x, y \in \mathbb{F}$, $x \sim y$ whenever $x = y$ or $xy = 1$. Note that \sim is an equivalence relation.

Lemma 14.2. *The following (i)–(iii) hold:*

- (i) $\overline{A} = 0$ if and only if $b \sim c$;
- (ii) $\overline{B} = 0$ if and only if $c \sim a$;
- (iii) $\overline{C} = 0$ if and only if $a \sim b$.

Proof. (i) First assume that $a \sim q^{d+1}$. We show $[d+1]_q \neq 0$. Suppose $[d+1]_q = 0$. Then $1 = q^{2d+2}$, which implies $d \geq 1$. Also $a = a^{-1}$, which implies $\theta_0 = \theta_d$. This contradicts the fact that $\{\theta_i\}_{i=0}^d$ are mutually distinct. Therefore $[d+1]_q \neq 0$. Now by (22), (23) we find $\overline{A} = 0$ iff $b \sim c$. Next assume that $a \not\sim q^{d+1}$. Using (15), (24) we obtain $\overline{A} = 0$ iff $b \sim c$.

(ii), (iii) Similar to the proof of (i) above. \square

Corollary 14.3. *The following are equivalent:*

- (i) $\overline{A} = \overline{B} = \overline{C} = 0$;
- (ii) a, b, c are mutually similar;
- (iii) the Leonard triple A, B, C is modular in the sense of Definition 4.7.

Proof. (i) \Leftrightarrow (ii) By Lemma 14.2.

(i) \Rightarrow (iii) Recall that $WA = AW$ and $W'B = BW'$. Since $\overline{A} = 0$ we have $W^{-1}BW = C$. Since $\overline{B} = 0$ we have $C = W'A(W')^{-1}$. So $W^{-1}BW = W'A(W')^{-1}$. Now in the sense of [9, Definition 1.2], the pair A, B is a spin Leonard pair with associated Boltman pair $W^{-1}, (W')^{-1}$. Now by [9, Theorem 1.6], the Leonard triple A, B, C is modular.

(iii) \Rightarrow (ii) By [8, Lemma 9.3] the Leonard triples A, B, C and B, C, A are isomorphic. The scalars a, b, c are mutually similar by Note 5.9 and Lemma 5.11. \square

Assumption 14.4. For the rest of this section, assume $\overline{A} = \overline{B} = \overline{C} = 0$. By Corollary 14.3 the scalars a, b, c are mutually similar. Inverting the eigenvalue orderings $\{\theta_i\}_{i=0}^d$, $\{\theta'_i\}_{i=0}^d$, $\{\theta''_i\}_{i=0}^d$ as necessary, we will assume that $a = b = c$.

Lemma 14.5. *Under Assumption 14.4 we have $\theta_i = \theta'_i = \theta''_i$ for $0 \leq i \leq d$.*

Proof. By (12) and $a = b = c$. \square

Recall the maps W, W', W'' from Definition 8.1.

Definition 14.6. Define $P = W'W$.

Lemma 14.7. *Under Assumption 14.4 we have*

$$P^{-1}AP = B, \quad P^{-1}BP = C, \quad P^{-1}CP = A.$$

Proof. Use Proposition 9.2 and Definition 14.6. \square

Lemma 14.8. *Under Assumption 14.4 the following holds for $0 \leq i \leq d$:*

$$P^{-1}E_iP = E'_i, \quad P^{-1}E'_iP = E''_i, \quad P^{-1}E''_iP = E_i.$$

Proof. Use (3) along with Lemmas 14.5, 14.7. □

Lemma 14.9. *Under Assumption 14.4 we have*

$$P^{-1}WP = W', \quad P^{-1}W'P = W'', \quad P^{-1}W''P = W.$$

Proof. Use Definition 8.1 and Lemma 14.8 and $a = b = c$. □

Proposition 14.10. *Under Assumption 14.4 the element P is equal to each of*

$$W'W, \quad W''W', \quad WW''.$$

Proof. In the equations $P^{-1}W'P = W''$ and $P^{-1}W''P = W$, eliminate P using $P = W'W$. □

Corollary 14.11. *Under Assumption 14.4 we have $P^3 = a^{-3d}q^{d(d-1)}I$.*

Proof. Use Theorem 13.5, Proposition 14.10, and $a = b = c$. □

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